Calibrating to Market Data – Getting the Model into Shape

Tutorial on Reconfigurable Architectures in Finance

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What is calibration?

Focus of this presentation: Option valuation within the stochastic volatility model of Heston [1993].

In this context, calibrating a model to market data means determining its endogenous parameters in such a way it describes observable market data as good as possible.

Agenda

Financial market and European call
Calibration: Theoretical aspects and a pseudo algorithm
Relevant financial products
What to do with a calibrated model?
Conclusion
Financial market

As financial market we consider the model defined in Heston [1993].

\[ dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^S, \quad S_0 = s_0 > 0, \]
\[ dV_t = \kappa (\theta - V_t) dt + \eta \sqrt{V_t} dW_t^V, \quad V_0 = v_0 > 0, \]

with interest rate \( r > 0 \) and correlation \( \rho \in [-1, 1] \) between the Brownian motions.

\( V_t \) is a mean-reverting stochastic process with long-term mean \( \theta > 0 \), speed of the mean-reversion \( \kappa > 0 \) and \( \eta > 0 \) reflects the volatility of the variance process.

The change in the asset price \( dS_t \) consists of a deterministic and a random component which is scaled by a random size \( \sqrt{V_t} S_t \).

The model is very popular among practitioners because it extends the very restrictive assumption of a constant volatility in the Black-Scholes model [1973] and still features a (semi-)analytic solution for a European call.
What is a European call?

A European call is a contract that gives the buyer the right (but not the obligation) to buy an underlying asset at a specified price $K$ on a specified date $T$ in the future.

Example: Assume the current stock price to be $S_0 = 100$. We expect the stock to rise and buy a call with $K = 110$ that lives for one year ($T = 1$). If in one year the stock price is $S_1 = 130$ (or $S_1 = 90$) and we are allowed to buy the stock for $K = 110$, we realise a profit of 20 (or 0).

The payoff of a European call in $T$ is $\max(0, S_T - K)$.

Since the option offers a right to buy the stock, it should have a price. How much should it cost today?
Price of a European call in the Heston model

In the Heston model, the price $X(K, T, \nu_0, \kappa, \theta, \eta, \rho)$ of a European call is given as

$$X(K, T, \nu_0, \kappa, \theta, \eta, \rho) = S_0 Q_1 - K e^{-rT} Q_2$$

with

$$Q_j = \frac{1}{2} + \frac{1}{\pi} \left( \frac{e^{-rT}}{S_0} \right)^{2-j} \int_0^\infty \Re \left( \frac{\phi(u - i(2-j))}{iu \exp(iu \log(K))} \right) du,$$

where $\phi(u) = \exp(A_1(u) + A_2(u) + A_3(u))$ and

$$A_1(u) = iu (\log(s_0) + rT),$$

$$A_2(u) = \frac{\theta \kappa}{\eta^2} \left( (\kappa - \rho \eta iu - h(u)) T - 2 \log \left( \frac{1 - g(u) e^{-h(u)T}}{1 - g(u)} \right) \right),$$

$$A_3(u) = \frac{\nu_0 (\kappa - \rho \eta iu - h(u)) \left( 1 - e^{-h(u)T} \right)}{\eta^2 \left( 1 - g(u) e^{-h(u)T} \right)},$$

$$g(u) = \frac{\kappa - \rho \eta iu - h(u)}{\kappa - \rho \eta iu + h(u)}, \quad h(u) = \sqrt{(\rho \eta iu - \kappa)^2 + \eta^2 (\kappa + u^2)}.$$

Although the formula looks very complicated it can be calculated efficiently!
How do we determine the relevant parameters of the model?

On the market, many institutions (banks) offer European calls on various assets like single stocks or indices.

These options are widespread and traded frequently (they are liquid), so supply and demand will clear at a fair price.

This information (i.e. the option prices) can be used to determine the model parameters. I.e. we try to find model parameters in such a way that the theoretical model values and the observed market prices are as close as possible. This process is called calibrating the model to market data.
How to calibrate the model?

A simple and popular calibration algorithm utilises the least-squares approach.

Let \( X(K_i, T_i)^{\text{market}}, i = 1, \ldots, N \) be \( N \) observable market prices of European calls with strikes \( K_i \) and maturities \( T_i \).

Let \( X(K_i, T_i, \Theta) \) be model prices of the corresponding options (i.e. for the same set of strikes and maturities) calculated using \( \Theta = (\nu_0, \kappa, \theta, \eta, \rho) \).

The calibration problem then reads as

\[
\min_{\Theta} \sum_{i=1}^{N} \omega_i \left( X(K_i, T_i)^{\text{market}} - X(K_i, T_i, \Theta) \right)^2. \tag*{=:f(\Theta)}
\]

For practical applications, some market prices might have different importance to the calibration. The weights \( \omega_i \) allow to stress or damp the impact of particular information.
A pseudo algorithm and important remarks

1. Choose initial parameter $\Theta_0$ and determine $f_0 = f(\Theta_0)$. Save both as $\Theta^*$ and $f^*$.
2. Until termination conditions (time, function calls, tolerance, ...) are hit, do
   1. Let an optimization algorithm choose a candidate $\Theta_j$ with $f_j = f(\Theta_j)$.
   2. If $f_j < f^*$ overwrite $\Theta^*$ and $f^*$ with $\Theta_j$ and $f_j$.
3. Return $\Theta^*$ and $f^*$.

As optimization algorithm, one can either apply

- deterministic methods, which are fast but restricted to a local minimum and thus must be restarted with different $\Theta_0$ or
- stochastic algorithms, which perform a global search but are very time consuming.

Typically the objective function has several local minima and the model calibration is performed at least daily for practical applications.
Are other (more complex) derivatives relevant as well?

Alternatively one could ask: “Why is the presence of an analytic formula for a European call so important?”

Pricing more complex products often requires advanced numerical schemes like

- **Monte Carlo methods**, which form a very general approach but face slow convergence → time consuming, or
- **Finite difference schemes**, which are flexible for many payoffs but depend heavily on the underlying grid. Accurate pricing requires a substantial amount of grid points → expensive.

Since we need to compute the objective function many times when calibrating, only products with analytic solutions are taken into account.

“Therefore, closed pricing formulae are the basis of a convenient model calibration.”

Still calibration is time consuming!

Assume a naive optimizing algorithm that

- takes $\Theta_0 = (v_0, \kappa, \theta, \eta, \rho)$ as input,
- assigns 4 test values (e.g. $\Theta_0 \pm 10\%$ and $\Theta_0 \pm 20\%$) to each component and
- determines the objective function for each of these combinations.

If further the market data consists of 100 observed call prices, we have to evaluate the formula $4^5 \times 100 = 102,400$ times. If each function call takes only 0.25 seconds, the total calibration takes more than 7 hours!

Model calibration has to be done for each stock or index!

On November 29th, 2013 Thomson Reuters (a financial data provider) listed 761 European call options on the DAX (“Deutscher Aktien Index”) for 12 different maturities and 130 different strike prices.

“The price to pay for more realistic models is the increased complexity of model calibration.” in Sana Ben Hamida and Rama Cont. Recovering volatility from option prices by evolutionary optimization. *Journal of Computational Finance*, 8(3) pp. 43-76 (2005).
So we found the parameters! What do we do now?

Obviously the Heston model is not intended to calculate European call prices.

Once its parameters are found, we can employ Monte Carlo methods or finite differences schemes and price more complex derivatives that do not have analytic solutions.

Reasons for exotic derivatives:

- Complex derivatives often have lower premiums compared to their corresponding simple options.
- Since exotics typically have special features, they can help investors to achieve specific objectives in risk management.
- Low interest rates and poor economic environments “force” investors to invest into exotics to achieve enhanced yields.
Conclusion: So the faster the better?

Fast computations are always wanted, but being mathematically correct is important as well.

In the literature (and in the original article), the characteristic function of the Heston model is stated in an alternative and correct form.

However, when numerically computing call prices, it turned out to be rather useless due to the implementation of the complex logarithm in many software tools.
What is the conclusion?

Financial institutions will pay more and more attention to computing power. On the other hand, if financial mathematics theory can tell the exact price, one does not have to compute at all. → Combination of both offers a promising approach!

Indeed, banks already employ “supercomputer”,

- [www.risk.net](http://www.risk.net) (*February 6rd, 2012*) “Need for speed: banks explore FPGAs for portfolio modelling”
  - “. . . running complex simulations for each of the relevant portfolios, under each of the required scenarios – a process that takes vast amounts of computing power and hours to complete, . . .”
  - “Piling on more computing cores gets very expensive.”

  - “JPMorgan is using a Maxeler supercomputer to measure risk . . . .”
  - “With the supercomputer, . . . , the investment bank can calculate complex scenarios which used to take hours in just a few minutes.”
The end

Thank you very much for your attention!